Homework for the summer school of Amplitudes 2021

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Please hand in your solution to me personally, or send the electronic version to `wuzihao@mail.ustc.edu.cn`.

I. UT INTEGRAL, SYMBOLS AND POLYLOGARITHMS

Problem I.1. *(Massless double box DE integration). In this exercise, we finish the computation of the massless four-point double box integrals. Use the notation in the lectures.*

1. *Find the analytic boundary values of the UT integrals at $x = 1$ up to weight 4. (Hint) Numerically, by pySecdec, the UT integral I_1 at $x = 1$ has the expansion*

$$\epsilon^4 I_1|_{x=1} = 2.25000 - 19.3280\epsilon^2 - 55.6127\epsilon^3 - 62.5477\epsilon^4 + O\left(\epsilon^5\right) \tag{1}$$

   Check your analytic expression with this.

2. *Analytically solve the canonical DE with the boundary values, up to weight 4. (Hint) Numerically, by pySecdec, the UT integral I_1 at $x = -3 - i\delta$, $\delta > 0$, has the expansion,*

$$\begin{aligned}\epsilon^4 I_1|_{x=-3-i\delta} = 2.25000 - (2.19723 - 6.28319i)\epsilon - 19.328\epsilon^2 - (9.45502 + 30.7348i)\epsilon^3 \\ +(50.4616 - 26.8575i)\epsilon^4 + O\left(\epsilon^5\right)\end{aligned} \tag{2}$$

   Check your analytic expression with this. Note that this is a physical point, and the error of pySecdec may be large.

3. *For your analytic solution up to weight 4, compute the symbol of the solution.*

4. *Without the analytic solution, directly integrate the canonical DE to a symbol solution. Compare your result with the answer to the previous question.*

Problem I.2. *(One-loop box with internal mass)*

1. *Given the alphabet $\{W_1, \ldots, W_8\}$ defined in the lecture, derive the canonical differential equation with numerical intepolation.*

2. *Prove that the symbol*

$$S[\frac{\beta_u - 1}{\beta_u + 1}, \frac{\beta_{uv} - \beta_u}{\beta_{uv} + \beta_u}] + S[\frac{\beta_v - 1}{\beta_v + 1}, \frac{\beta_{uv} - \beta_v}{\beta_{uv} + \beta_v}] \tag{3}$$

with $\beta_u = \sqrt{1+u}$, $\beta_v = \sqrt{1+v}$ *and* $\beta_{uv} = \sqrt{1+u+v}$, *is integrable. Prove that each individual term is not integrable.*

Problem I.3. *(Heavy quark effective theory) Consider the two-loop integral family with,*

$$\begin{aligned} &D_1 = 2l_2 \cdot v_1 - 1, \quad D_2 = 2l_2 \cdot v_2 - 1, \quad D_3 = (l_1 - l_2)^2, \\ &D_4 = 2l_1 \cdot v_1 - 1, \quad D_5 = 2l_1 \cdot v_2 - 1, \quad D_6 = l_1^2, \quad D_7 = l_2^2 \end{aligned} \tag{4}$$

with l_1, l_2 *the loop momenta, and* v_1, v_2 *the external momenta for heavy quark velocity:*

$$v_1^2 = v_2^2 = 1, \quad v_1 \cdot v_2 \equiv \frac{1}{2}\left(x + \frac{1}{x}\right) \tag{5}$$

Note that as HQET, there are linear propagators here.

1. *Find the master integrals of the sector* $(1,1,1,1,1,1,0)$ *and its subsectors. Derive the differential equation for the these integrals.*
2. *Use the package* LIBRA *or* EPSILON *to transform the differential equation to canonical differential equation.*
3. *Does the previous transformation find a UT basis? If not, please fix it to get a UT basis. Hint: you may explicitly compute the lowerest master integral or consider the leading singularity of the integral* $G[1,1,1,1,1,1,0]$.
4. *find the boundary values of the UT basis at* $x = 1$. *Hint: if the UT basis you found in the previous question contains a factor,*

$$\frac{1}{x-1} \tag{6}$$

   in the definition, then you need to compute an asymptotic expansion of the master integrals at $x \to 1$. *Alternatively, you may find a better UT basis without the factor* $1/(x-1)$.

5. *With the boundary value at* $x = 1$, *analytically compute the UT basis up to the weight-4 order. Hint: at* $x = 1/2$, *from pySecdec we have,*

$$e^{2\epsilon\gamma} G[1,1,1,1,1,1,0]|_{x=1/2} = (0.106767)\epsilon^{-2} - (0.202461)\epsilon^{-1} + 2.00144 + O(\epsilon) \tag{7}$$

   Check your analytic expression against the numeric result.

Problem I.4. *(One-loop three-mass box). Consider the one-loop family with the inversed propagators,*

$$D_1 = \ell_1^2, \quad D_2 = (\ell_1 - p_1)^2, \quad D_3 = (\ell_1 - p_1 - p_2)^2, \quad D_4 = (\ell + p_4)^2 \tag{8}$$

with

$$p_1^2 = m_1^2, \quad p_2^2 = m_2^2, \quad p_3^2 = m_3^2, \quad p_4^2 = 0, \quad (p_1+p_2)^2 = s, \quad (p_2+p_3)^2 = t \tag{9}$$

1. *Find the master integrals of this family.*
2. *Find the UT basis of this family. You are going to get square roots. Enjoy.*
3. *Find the canonical DE and decompose it to get the symbol letters.*
4. *On the symbol level, integrate the canonical DE to the weight* 3.

Problem I.5. *(One-loop pentagon). Consider the one-loop family with the inversed propagators,*

$$D_1 = \ell_1^2, \quad D_2 = (\ell_1 - p_1)^2, \quad D_3 = (\ell_1 - p_1 - p_2)^2, \quad D_4 = (\ell - p_1 - p_2 - p_3)^2, \quad D_5 = (\ell + p_5)^2 \tag{10}$$

with

$$p_i^2 = 0, \quad i = 1, \ldots, 5, \quad p_i \cdot p_j = \frac{s_{ij}}{2} \tag{11}$$

Choose s_{12}, s_{23}, s_{34}, s_{45}, s_{15} as kinematic variables.

1. *Find the master integrals of this family.*
2. *Find the UT basis of this family. You are going to get one square root which is ϵ_{1234}.*
3. *Find the canonical DE and decompose it to get the symbol letters.*
4. *On the symbol level, integrate the canonical DE to the weight* 3.

II. COMPUTATIONAL ALGEBRAIC GEOMETRY

Problem II.1. *(Impression of algebraic varieties)*

1. *(Heart curve) Consider the parametric curve in the x-y plane,*

$$(x, y) = \big(16\sin^3(t), 13\cos(t) - 5\cos(2t) - 2\cos(3t) - \cos(4t)\big) \tag{12}$$

   The plot is given in Fig. 1. Prove this curve is algebraic and find a polynomial $F(x,y)$ such that $F(x,y) = 0$ defines this curve.

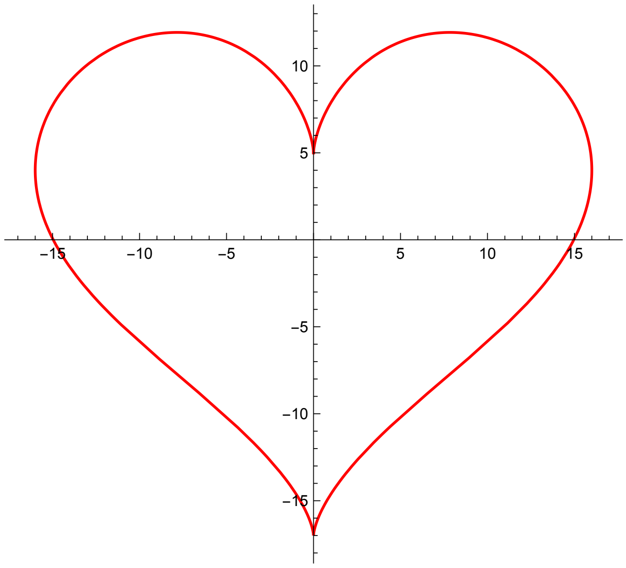

FIG. 1: Heart

2. *(Group Law of the elliptic curve). Consider the Weierstrass elliptic curve* E*,*

$$E: \quad y^2 = x^3 + ax + b \tag{13}$$

with $\Delta = -16(4a^3 + 27b^2) \neq 0$.

a) Consider two generic points $P = (x_P, y_P)$ *and* $Q = (x_Q, y_Q)$ *on this curve and* $P \neq Q$*. The straight line connecting* P *and* Q*, intersects* E *at the point* R*. Write* R*'s coordinates as rational functions of* P*,* Q*'s coordinates.*

Then we formally define

$$P + Q + R = O \tag{14}$$

while O *denotes the point of infinity.*

b) Similary, for a genric point $P = (x_P, y_P)$ *on the curve. Let* l *be a straight line tangent to* E *at* P*.* l *intersects* E *at* R*. Write* R*'s coordinates as rational functions of* P*.*

In this case, we formally define

$$P + P + R = O, \quad \text{or} \quad 2P + R = O \tag{15}$$

where O *is the point of infinity. We also define* $-P$ *as the point* $(x_P, -y_P)$*. Treat* O *as the zero element for "+", and thus the curve* E *forms an Abelian group. The group structure is crucial for the study of elliptic curve.*

c) Explicitly, let E be the elliptic curve $y^2 = x^3 + 17$. Check that the points $P_1 = (-2, 3)$ and $P_2 = (2, 5)$ are on the curve. With the group law, expliclity compute

$$2P_1, \quad P_1 + P_2 \,. \tag{16}$$

Define $P_3 = (52, 375)$. Find two integers n_1 and n_2, such that

$$n_1 P_1 + n_2 P_2 = P_3 \tag{17}$$

This exercise is related to the famous theorem of Mordell-Weil.

3. *(Veronese and Segre Map) A Veronese map from $\mathbb{P}^2 \to \mathbb{P}^5$ is,*

$$\phi_V : \quad (x_0, x_1, x_2) \mapsto (x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2) \tag{18}$$

The image of this map, which is a projective variety (Veronse variety), is denoted as V_1. Name the coordinates of $\mathbb{P}^5$ as $[z_0, \ldots, z_5]$ and it coordinate ring as $R = [z_0, \ldots, z_5]$. Compute the ideal $\mathcal{I}(V_1)$.

A Segre map from $\mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^5$ is,

$$\phi_S : \quad (w_0, w_1, w_2, y_0, y_1) \mapsto (w_0y_0, w_0y_1, w_1y_0, w_1y_1, w_2y_0, w_2y_1) \tag{19}$$

The image of this map, is also a projective variety V_2 (Segre variety). Compute the ideal $\mathcal{I}(V_2)$.

What is $V_1 \cap V_2$? Hint: consider the primary decomposition.

Problem II.2. *(Some trivial facts)*

1. *Consider a polynomial ring $R = \mathbb{Q}[x_1, \ldots x_n]$. I and J are ideals of R and $I + J = R$. Prove that $IJ = I \cap J$.*

2. *Prove that $\mathbb{C}^n - \{0\}$ is not an affine variety.*

3. *Let $R = \mathbb{Q}[x_1, \ldots x_n]$ and I is a prime ideal of R. Suppose that R/I is a finite-dimensional linear space over $\mathbb{Q}$. Prove that R/I is a field.*

Problem II.3. *(Zero-dimensional ideal) Consider the Baxter QQ relation for the XXZ model with twisted boundary condition. The equation, after the simplification, reads*

$$\begin{gathered}\{-5s_2 - 12, -512s_0^2 - 16800s_1s_0 + 29008s_0 + 42875s_1^2 - 202125s_1 + 227500,\\ -42875s_1^3 + 245000s_1^2 + 2192s_0s_1 - 463925s_1 - 3312s_0 + 292860,\\ 560s_0s_1^2 - 1600s_0s_1 + 1225s_1 + 1008s_0 - 2412\}\end{gathered} \tag{20}$$

with s_0*,* s_1 *and* s_2 *as the variables. There are five solutions for this equation, namely* $p_1, \ldots p_5$*. The transfer function, after the simplification, reads,*

$$T = \frac{-1225s_1^2 - 816s_0s_1 + 2940s_1 + 816s_0 - 10176}{1728} \tag{21}$$

Analytically compute

$$\sum_{i=1}^{5} \left(T^{128}\big|_{p_i} \right) \tag{22}$$

The result would be the partition function of a two-dimensional lattice model.

Problem II.4. *Consider*

$$f_1 = x^2 - y^2 - z^2 + 1, \quad f_2 = x - 3y + zx, \quad f_3 = x^2 - xy + z^2 - 1 \tag{23}$$

Compute the (Grothendieck) multivarite residue of

$$\frac{x}{f_1^2 f_2 f_3} \tag{24}$$

at $(x, y, z) = (0, 0, 1)$ *for the small contour* $|f_1| = |f_2| = |f_3| = \epsilon$*.*

Problem II.5. *(IBP from Syzygies) This problem is about the IBP reduction for the two-loop massless double box integrals with the syzygy approach [1]. Nowadays, it is much easier to do a syzyg IBP reduction with the Baikov representation [2], but here we consider the original syzygy method in the momentum space, since the picutre is clear.*

The double box integral family has the form,

$$G[m_1, \ldots, m_9] = \int \frac{d^D l_1 d^D l_2}{(i\pi^{D/2})^2} \frac{1}{D_1^{m_1} \ldots D_9^{m_9}} \tag{25}$$

where D_i*'s are defined in the lecture.*

1. *(IBP on the maximal cut). We consider the IBPs on the maximal cut, with the method in the ref. [1]. Focus on the sector* $(1, 1, 1, 1, 1, 1, 1, 0, 0)$*. An IBP has the form,*

$$0 = \int \frac{d^D l_1 d^D l_2}{(i\pi^{D/2})^2} \left(\frac{\partial}{\partial l_1^\mu} \frac{v_1^\mu}{D_1^{m_1} \ldots D_9^{m_9}} + \frac{\partial}{\partial l_2^\mu} \frac{v_2^\mu}{D_1^{m_1} \ldots D_9^{m_9}} \right). \tag{26}$$

We use the following 9 *quantities as free variables,*

$$\mathcal{X} = \Big\{ (l_1 \cdot p_1), \quad (l_1 \cdot p_2), \quad (l_1 \cdot p_4), \quad (l_2 \cdot p_1), \quad (l_2 \cdot p_2), \quad (l_2 \cdot p_4), \quad l_1^2, \quad (l_1 \cdot l_2), \quad l_2^2 \Big\}. \tag{27}$$

Write the inverse propagators $D_1, \ldots, D_9$ as linear functions of these variables.

The vector v_1 and v_2 are parametrized as,

$$\begin{aligned} v_1^\mu &= a_1 p_1^\mu + a_2 p_2^\mu + a_3 p_4^\mu + a_4 l_1^\mu + a_5 l_2^\mu \\ v_2^\mu &= a_6 p_1^\mu + a_7 p_2^\mu + a_8 p_4^\mu + a_9 l_1^\mu + a_{10} l_2^\mu \end{aligned} \tag{28}$$

where $a_1, \ldots, a_{10}$ should be polynomials in $\mathcal{X}$.

Consider the syzygy relations,

$$\left(\sum_{i=1}^{2} v_i^\mu \frac{\partial D_j}{\partial l_i^\mu} \right) + b_j D_j = 0, \quad j = 1, \ldots 7 \tag{29}$$

There are 7 such equations. Here $b_1, \ldots, b_7$ should also be polynomials in $\mathcal{X}$. Explicitly write the syzygy relations as,

$$A(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, b_1, b_2, b_3, b_4, b_5, b_6, b_7)^T = 0\,. \tag{30}$$

where A is a 7×17 matrix.

Apply Singular or the interface to Singular for Mathematica, to solve this syzygy problem. The computation should be fast, i.e., in seconds. Call the solutions $v^{(1)}, \ldots, v^{(N)}$. Do not worry if N is large. Instead of trimming the syzygy generator set, we just naively use some linear algebra tricks to get simple IBP sets.

First generate the IBPs with all N vectors with

$$m_1 = \ldots = m_7 = 1, \quad m_8 = 0, \quad m_9 = 0 \tag{31}$$

We call it the basic IBP for a vector. If a vector's basic IBP has no integral in this sector $(1,1,1,1,1,1,1,0,0)$, drop it since this vector would provide sub-sector IBPs.

Define $H = 6$, the highest degree for ISPs. For every surviving vector's basic IBP, denote the D_8, D_9's total degree as h, ($h \geq 0$). Perform a seeding for this vector with all possible values of m_8 and m_9,

$$m_8 \leq 0, \quad m_9 \leq 0, \quad -(H-h) \leq m_8 + m_9 \leq 0, \tag{32}$$

and you will get several IBPs for this vector.

Collect all IBPs you have from the previous step. Then set all integrals which are not on the sector $(1,1,1,1,1,1,1,0,0)$ as zero. This is the maximal cut. You will get hundreds of IBPs in 28 integrals. Convince yourself that these integrals therein have no double propagators.

Find a linearly independent set of IBPs with numerical linear algebra computations. Now you have 26 *independent IBPs in* 28 *integrals. Define an integral ordering and do a Gaussian elimination on these IBPs. This can be easily done with* RowReduce *in Mathematica, within seconds.*

You result should be the reduction of all ISP degree ≤ 6 *integrals of this sector to*

$$G[1,1,1,1,1,1,1,0,0], \quad G[1,1,1,1,1,1,1,-1,0] \tag{33}$$

on the maximal cut.

2. (257*-cut). We want the full IBP instead of the maximal cut IBP. Frequently, we use some spanning cuts to reconstruct the full IBPs to speed up the computations.*

   For the double box, one spanning cut is 257*, that means any integral in a sector* S *with*

$$a_2 = 0, \ or \quad a_5 = 0, \ or \quad a_7 = 0 \tag{34}$$

   is set to zero. We need to consider $2^4 = 16$ *sectors for this cut:*

$$\begin{gathered}\{1,1,1,1,1,1,1,0,0\},\{1,1,1,1,1,0,1,0,0\},\{1,1,1,0,1,1,1,0,0\},\{1,1,0,1,1,1,1,0,0\}\\ \{0,1,1,1,1,1,1,0,0\},\{1,1,1,0,1,0,1,0,0\},\{1,1,0,1,1,0,1,0,0\},\{1,1,0,0,1,1,1,0,0\},\\ \{0,1,1,1,1,0,1,0,0\},\{0,1,1,0,1,1,1,0,0\},\{0,1,0,1,1,1,1,0,0\},\{1,1,0,0,1,0,1,0,0\},\\ \{0,1,1,0,1,0,1,0,0\},\{0,1,0,1,1,0,1,0,0\},\{0,1,0,0,1,1,1,0,0\},\{0,1,0,0,1,0,1,0,0\}\end{gathered} \tag{35}$$

   For the IBP of the cut 257*, we have the following instructions:*

   Compute the syzygies for the 16 *sectors. For each sector, we impose the condition that there is no double propagator. For example, for the sector* $\{0,1,0,0,1,1,1,0,0\}$*,*

$$\left(\sum_{i=1}^{2} v_i^\mu \frac{\partial D_j}{\partial l_i^\mu}\right) + b_j D_j = 0, \quad j = 2,5,6,7 \tag{36}$$

   For every sector, perform a seeding for the ISPs. For the top sector $\{1,1,1,1,1,1,1,0,0\}$*, we use* $H = 5$*. For other sectors, we use* $H = 4$*. (These values are determined by the trial and error.) For example, for a vector associated with the sector* $\{0,1,0,0,1,1,1,0,0\}$*, we use*

$$\begin{gathered} m_1 \leq 0, \quad m_3 \leq 0, \quad m_4 \leq 0, \quad m_8 \leq 0, \quad m_9 \leq 0, \\ -(4-h) \leq m_1 + m_3 + m_4 + m_8 + m_9 \leq 0. \end{gathered} \tag{37}$$

where h ($h \geq 0$) is the ISP degree of the basic IBP.

For each sector, collect the IBPs. Here we use the classic "tail mask" trick. Treat the integrals on the sector as variables, and all other integrals as constants. Use linear algebra to find an independent IBP set for each sector.

Each sector's IBP system would be quite small, i.e., with fewer than 300 IBPs. Then use Mathematica (not the command "RowReduce"!), Singular, Fermat or FiniteFlow to do a Gaussian Elimination. Each Gaussian elimination should be fast, finished within a minute.

Combine the Gaussian elimination results of all 16 sectors, to reduce $G[1,1,1,1,1,1,1,0,-5]$ as a linear combination of,

$$G[1,1,1,1,1,1,1,-1,0], G[1,1,1,1,1,1,1,0,0], G[1,1,1,0,1,0,1,0,0], G[1,1,0,1,1,0,1,0,0],$$
$$G[0,1,1,0,1,1,1,0,0], G[0,1,0,1,1,1,1,0,0], G[0,1,0,0,1,0,1,0,0] \tag{38}$$

This step is called the back-substitution. Note that you have to simplify the immediate expressions in combining the Gaussian elimination results, to speed up the computation. Here we do not consider the symmetries.

If you do it properly, this step can be done within seconds in Mathematica.

[1] J. Gluza, K. Kajda, and D. A. Kosower, Phys.Rev. **D83**, 045012 (2011), 1009.0472.

[2] K. J. Larsen and Y. Zhang, Phys. Rev. D **93**, 041701 (2016), 1511.01071.